A NOTE ON KÄHLER MANIFOLDS WITH ALMOST NONNEGATIVE BISECTIONAL CURVATURE

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Abstract. In this note we prove the following result: There is a positive constant $\varepsilon(n,\Lambda)$ such that if M^n is a simply connected compact Kähler manifold with sectional curvature bounded from above by Λ, diameter bounded from above by 1, and with holomorphic bisectional curvature $H \geq -\varepsilon(n,\Lambda)$, then M^n is diffeomorphic to the product $M_1 \times ... \times M_k$, where each M_i is either a complex projective space or an irreducible Kähler-Hermitian symmetric space of rank ≥ 2. This resolves a conjecture of F. Fang under the additional upper bound restrictions on sectional curvature and diameter.

1. Introduction

In a recent paper [PT] Petersen and Tao give a classification of simply connected compact manifold with almost quarter-pinched curvature using the Ricci flow method. Inspired by their work, in this short note we will try to classify simply connected compact Kähler manifolds with almost nonnegative bisectional curvature.

First recall (cf. Fang [F]) that a compact complex manifold M has almost nonnegative bisectional curvature if given any $\varepsilon > 0$, there is a Hermitian metric g on M with bisectional curvature H satisfying $H \cdot diam(M_g)^2 \geq -\varepsilon$.

Our main result is the following

Theorem There is a positive constant $\varepsilon(n,\Lambda)$ such that if M^n is a simply connected compact Kähler manifold with sectional curvature bounded from above by Λ, diameter bounded from above by 1, and with holomorphic bisectional curvature $H \geq -\varepsilon(n,\Lambda)$, then M^n is diffeomorphic to the product $M_1 \times ... \times M_k$, where each M_i is either a complex projective space or an irreducible Kähler-Hermitian symmetric space of rank ≥ 2.

This improves earlier work of Fang [F].

When we impose an additional condition that the second Betti number $b_2(M^n) = 1$, we have the following

Corollary There is a positive constant $\varepsilon(n,\Lambda)$ such that if M^n is a simply connected compact Kähler manifold with sectional curvature bounded from above by Λ, diameter bounded from above by 1, holomorphic bisectional curvature $H \geq -\varepsilon(n,\Lambda)$, and with the second Betti number $b_2(M^n) = 1$, then M^n is diffeomorphic

1991 *Mathematics Subject Classification.* 53C44.

Key words and phrases. Kähler manifolds, almost nonnegative bisectional curvature, Ricci flow.

Partially supported by NSFC no.10671018.

to either the complex projective space or an irreducible Kähler-Hermitian symmetric space of rank ≥ 2.

This resolves a conjecture of Fang [F] under the additional upper bound restrictions on sectional curvature and diameter.

As in [PT] we also use the Ricci flow method in the proof of our theorem, which is given in the following section.

2. Proof of Theorem

We prove by contradiction. Suppose the result is not true. Then we can find a sequence of simply connected compact Kähler manifolds (M_i, g_i) of complex dimension n satisfying the following condition:

i) the bisectional curvature H_i of M_i satisfies $H_i \geq -\frac{1}{i}$, the sectional curvature K_i of M_i satisfies $K_i \leq \Lambda$, and the diameter $diam(M_i) \leq 1$;

ii) none of M_i is diffeomorphic to a product where each factor is a complex projective space or an irreducible Kähler-Hermitian symmetric space of rank ≥ 2.

Note that the sectional curvature of (M_i, g_i) is uniformly bounded from below by a constant λ. Then by Hamilton's work [H1] we can run the Kähler-Ricci flow $(M_i, g_i(t))$ for a fixed amount of time $[0, t_0] (t_0 > 0)$ with initial data (M_i, g_i), with the sectional curvature of $(M_i, g_i(t))$ uniformly bounded on this time interval.

Also note that by a result of Fang and Rong [FR], the injectivity radius of (M_i, g_i) is uniformly bounded away from zero. Then by Hamilton's compactness theorem for the Ricci flow [H3], $(M_i, g_i(t))$ sub-converge to a Kähler Ricci flow $(M, g(t)), t \in [0, t_0]$. The convergence is in the $C^{1,\alpha} (\alpha < 1)$ sense at $t = 0$, and is in the C^∞ sense for $0 < t \leq t_0$.

Then by virtue of Hamilton's work (see the proof of [H2, Theorem 4.3]; compare with [PT]), the bisectional curvature of $(M_i, g_i(t))$ is bounded from below by $-\frac{exp(Ct)}{i}$, where C is a constant depending only on the curvature bounds for $g_i(0)$. (Note that here we have used implicitly the work of Bando [B] (in complex dimension 3) and Mok [M](in any dimension) on preserving nonnegative bisectional curvature condition under the Kähler-Ricci flow.) So the limit $(M, g(t))$ has nonnegative bisectional curvature for $t > 0$. Then by Mok's theorem [M] M is diffeomorphic to a product, where each factor is a complex projective space or an irreducible Kähler-Hermitian symmetric space of rank ≥ 2. But for i sufficiently large, M_i is diffeomorphic to M, so M_i is also diffeomorphic to such a product, a contradiction, and we are done.

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